Bayesian Estimation of Crack Initiation Times from Service Data

R.A. Heller* and G.H. Stevens†
Virginia Polytechnic Institute and State University, Blacksburg, Va.

Lockheed C-130 "Hercules" aircraft have been periodically inspected during their service life, and growing cracks around rivet holes were recorded. This record has recently been used to determine the statistical distributions of crack initiation times and the distribution of initial crack sizes. Because inspection procedures are not 100% reliable, several small cracks will be missed during early inspections. When crack initiation times are calculated from such cracks by backward extrapolation of the growth relation, the resulting distribution of crack initiation times will indicate a preponderance of short times to crack initiation. If however, such distributions are combined with the reliability of the inspection procedure, the statistical distribution of missed initiation times can be estimated. The method used is based on Bayes' theorem which permits the calculation of the "prior" distribution (initiation times before inspection) from a knowledge of the "posterior" distribution (initiation times obtained from the inspection) and a "likelihood function" (reliability of the inspection procedure). The results indicate that during an early inspection a large percentage of initiation times will be missed and that the fraction of located initiation times increases during later inspections.

Nomenclature

A,a	= crack length at time of inspection; random variable and given value
a_i	= initial crack length
$C,C^*,\bar{C}^*$	= crack propagation parameters
da/dt	= crack growth velocity
dN/dt	= rate of cycling
$D,\bar{D}$	= the event that a crack or an initiation time is detected; is not detected
$f(1/r)$	= crack propagation correction factors
$f_X(x)$	= probability density functions of the random variable X
$F_X(x)$	= cumulative probability function of the random variable X
i	= subscript
k_1,k_2,k_3,k_4	= normalizing constants
n,n^*	= crack propagation parameters
N	= number of cycles
$P[E_1 \mid E_2]$	= probability that event 1 will occur given that event 2 has already occurred
t	= inspection time
t_i,t_i'	= crack initiation time; parameter of Weibull distribution
T_i	= crack initiation time, random variable
$\alpha,\bar{\alpha}$	= shape parameters of Weibull distribution for crack initiation time, for inspection reliability
$\beta,\bar{\beta}$	= characteristic value of Weibull distribution for crack initiation time, for inspection reliability
$\delta(\cdot)$	= Dirac delta function
$\Delta\sigma$	= stress range

I. Introduction

In order to increase the safety and service life of aircraft structures, airplanes are subjected to rigorous scheduled inspections and maintenance. The Lockheed C-130 "Hercules" transport, in use by the U.S. Air Force for more than two decades has undergone such periodic inspections during which the size of cracks growing around rivet holes has been recorded.

The inspection data have recently been utilized[1,2] to determine the statistical distributions of initial crack sizes and of the times required for cracks to grow to an inspectable size. This was accomplished by backward extrapolation of a crack propagation law whose parameters were calculated from the inspection data. The importance of the "time to crack initiation" (time required for a crack to grow to an inspectable size) for the purposes of reliability analysis has been indicated by several authors.[3-5] To compute this important parameter the growth of cracks was monitored at identical locations on several aircraft operated by the same military command. The "power law" crack growth relation was then fitted to each growing crack and was eventually used to extrapolate backwards to the smallest inspectable size of 0.03 in. (0.76 mm) yielding the "time to crack initiation."

Such crack initiation times were statistically analyzed and three-parameter Weibull distributions were fitted to the data.[1,2] These distributions were derived from long-term records, in some instances covering 7000 flight hours. As a consequence most cracks have by this time grown to inspectable size and the statistical information deduced from them can be considered quite reliable.

Because inspection procedures are not 100% reliable it is an accepted fact that, particularly during early inspections, several small cracks will be missed. As a result the statistical distribution of crack sizes at any particular inspection will be biased towards large cracks. When crack initiation times are calculated from such cracks, by backward extrapolation of the crack growth relation, the resulting distribution of crack initiation times will indicate a preponderance of short times to crack initiation. If, however, such distributions are combined with the reliability of the inspection procedure, the statistical distribution of missed initiation times can be estimated.

The method used is based on Bayes' theorem which permits the calculation of the "prior" distribution (initiation times before inspection) from a knowledge of the "posterior" distribution (initiation times obtained from the inspection) and a "likelihood function" (probability of locating an initiation time, i.e., reliability of the inspection procedure).

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*Professor, Department of Engineering Science and Mechanics. Associate Fellow AIAA.

†Civil Engineer, Tennessee Valley Authority.

The likelihood function has been constructed from nondestructive examination (NDE) data[6] and is assumed to be a two-parameter Weibull function.

II. Determination of Crack Initiation Times

Experimental measurements of crack velocities indicate that the rate of crack propagation can be represented by a power function, the parameters of which depend on geometry, material properties, and the applied stress range.[7]

In order to determine the "time to crack initiation," t_i, a modified crack propagation rule was utilized in Refs. 1 and 2:

$$da/dt = C^* a^{n^*} \tag{1}$$

where

$$C^* = C[\Delta\sigma\sqrt{\pi}f(1/r)]^n dN/dt$$

with C and n material constants, $\Delta\sigma$ the stress range, $f(1/r)$ a geometric correction factor, dN/dt the rate of cycling, a the crack length at an inspection time, t, and $n^* = n/2$.

For an aluminum structure the material constant n has been determined from Refs. 7 and 8 as $n = 3$ or $n^* = 3/2$.

To determine the crack growth constant C^* for each growing crack individually, the average crack growth rate between two inspections was calculated as the difference in crack sizes divided by the inspection interval. It was evaluated at the average of the two crack sizes and at the midpoint of the interval. The values of C^* for the various growing cracks were averaged at each location to yields a $\bar{C}^*$.

The crack propagation rule, Eq. (1), was integrated

$$\int_{a_i}^{a} a^{-n^*} da = \bar{C}^* \int_{t_i}^{t} dt \tag{2}$$

with $a_i = 0.03$ in. (0.76 mm) the smallest inspectable crack size, to yield an a vs t relation

$$a = [\bar{C}^*(1-n^*)(t-t_i) + a_i^{(1-n^*)}]^{1/(1-n^*)} \tag{3}$$

which for $n^* = 3/2$ reduces to

$$a = \frac{1}{[1/\sqrt{a_i} - (\bar{C}^*/2)(t-t_i)]^2} \tag{4}$$

For a given inspection time, t, and an observed crack length, a, the crack initiation time, t_i, has been calculated for each growing crack. The t_i values for a particular location on the airplanes were then fitted with three-parameter Weibull distribution functions of the form

$$F_{T_i}(t_i) = 1 - \exp\left[-\left(\frac{t_i - t_i'}{\beta - t_i'}\right)^\alpha\right] \tag{5}$$

Because of the possible existence of flaws at time $t = 0$ this distribution function has a negative lower limit, t_i' and Eq. (5) is restricted to values for $t_i > 0$. Consequently the distribution of crack initiation times may be written as

$$F_{T_i}(t_i) = 0 \qquad \text{for} \quad t_i < 0 \tag{6}$$

$$F_{T_i}(0) = 1 - \exp\left[-\left(\frac{-t_i'}{\beta - t_i'}\right)^\alpha\right] \qquad \text{for} \quad t_i = 0 \tag{7}$$

and

$$F_{T_i}(t_i) = 1 - \exp\left[-\left(\frac{t_i - t_i'}{\beta - t_i'}\right)^\alpha\right] \qquad \text{for} \quad t_i > 0 \tag{8}$$

that is, the probability that the crack initiation time is zero has a fixed, nonzero value. The corresponding density function consists of a Dirac function, $\delta(t_i)$, at $t_i = 0$ with an area given by Eq. (7) and a truncated density function:[9]

$$f_{T_i}(t_i) = \left\{1 - \exp\left[-\left(\frac{t_i - t_i'}{\beta - t_i'}\right)^\alpha\right]\right\}\delta(t_i) + \frac{\alpha}{\beta - t_i'}\left(\frac{t_i - t_i'}{\beta - t_i'}\right)^{\alpha-1}\exp\left[-\left(\frac{t_i - t_i'}{\beta - t_i'}\right)^\alpha\right]$$

Such distribution functions have been fitted to the data and are presented in Refs. 1 and 2. Figure 1 shows such a distribution function for a typical rivet hole located in the corner of a cut out on the lower skin surface of the center wing box identified as location 76-92, for aircraft flying in the Tactical Air Command. The parameters used are as follows:

$$\bar{C}^* = 0.00537 \text{ in./in.}^{3/2}/\text{h}$$
$$\alpha = 9.5$$
$$\beta = 3500 \text{ h}$$
$$t_i' = -2000 \text{ h}$$

It should be noted that the function fits the data better for the more important shorter initiation times.

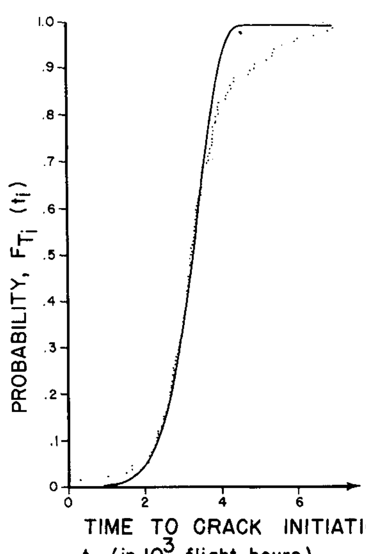

Fig. 1 Three-parameter Weibull distribution of crack initiation times.

III. Distributions of Detected and Undetected Cracks

The probability that an existing crack is detected is a function of a number of circumstances. These include: 1) the method of detection such as radiography, thermography, dye penetrant, ultrasonic or visual inspection, etc., and 2) the sensitivity and accuracy of the instrument or the alertness of the inspector. These conditions are influenced primarily by the size of the crack at the time of inspection.

As a consequence not all cracks are found and the probability, $P[D|a]$, that a crack is detected, given that its lenth is a, is a function of the crack size.

The method of inspection or the reliability of the procedure used for crack detection in C-130 airplanes are not available. Data can however be found in the literature on the probability of crack detection in aluminum using various ultrasonic techniques.[6] A set of data from Ref. 6 is shown in Fig. 2 with a Weibull-type probability function

$$P[D|a] = 1 - \exp\left[-\left(\frac{a}{\bar{\beta}}\right)^{\bar{\alpha}}\right] \tag{10}$$

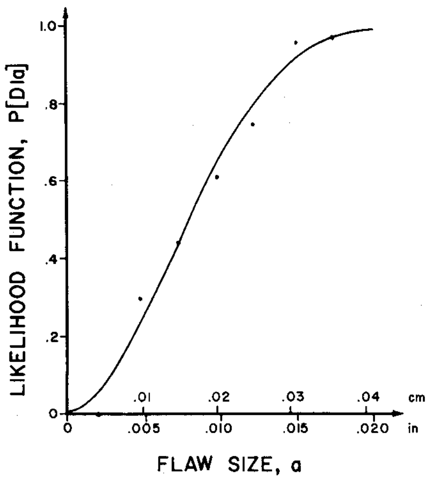

Fig. 2 Probability of crack detection, "likelihood function." Data from Ref. 6.

where $\bar{\alpha}=2.0$ and $\bar{\beta}=0.075$ in. (0.19 cm), fitted to it. Similar probability functions have been incorporated in wing failure analyses and have been reviewed by Eggwertz and Lindsjo.[10]

The C-130 inspection data indicate that cracks shorter than 0.03 in. (0.76 mm) have not been found. The assumed probability function used here predicts that a small fraction, 15%, of such cracks are located, while the probability of detecting a crack 0.2 in. in length is almost unity.

If the distribution of crack sizes prior to inspection is known, the distributions of detected and undetected cracks may be calculated utilizing the probability of detection function, Eq. (10) in conjunction with Bayes' theorem.[11,12]

The probability that a flaw of size, a, will be between x and $x+\mathrm{d}x$ given that the flaw is detected may be written as

$$P[a<A<(a+\mathrm{d}a)\,|\,D]$$

$$=\frac{P[D\,|\,a<A<(a+\mathrm{d}a)\,]P[a<A<(a+\mathrm{d}a)\,]}{P[D]} \tag{11}$$

where D stands for detection, $P[D\,|\,a<A<(a+\mathrm{d}a)\,]$ is the probability of detecting a flaw of size a [from Eq. (10)], $P[a<A<(a+\mathrm{d}a)\,]$ is the probability that the crack size is a, and $P[D]$ is the total probability of detection independently of size. Equation (11) may be expressed in terms of density functions as

$$f_A[a\,|D]=k_1P[D\,|a]f_A(a) \tag{12}$$

in which $f_A(a)$ is the density function of flaw sizes "prior" to detection, $f_A[a\,|D]$ is the density function of detected flaw sizes ("posterior density"), $P[D\,|a]$ defined earlier, is called the "likelihood function" in Bayes' theorem terminology, and $k_1=1/P[D]$ is a normalizing constant that will make the area under the $f_A(a\,|D)$ curve equal to unity.

In a similar manner the density function of undetected cracks may also be derived:

$$f_A[a\,|\bar{D}]=k_2P[\bar{D}\,|a]f_A(a) \tag{13}$$

where $\bar{D}$ stands for no detection, $P[D\,|a]=1-P[\bar{D}\,|a]$, and k_2 is a second normalizing constant. The above technique has been utilized by Davidson[9] in a slightly different form with exponential detectability and crack size distribution functions assumed.

The addition of Eqs. (12) and (13)

$$f_A(a\,|D)/k_1+f_A(a\,|\bar{D})/k_2=f_A(a) \tag{14}$$

indicates that the reciprocal normalizing constants, $1/k_1$ and $1/k_2$ represent the fractions of detected and undetected cracks.

Because the density function of cracks before inspection is usually unknown, the posterior function, $f_A[a\,|D]$, of detected cracks may be used to evaluate the prior, $f_A(a)$, and the density of undetected cracks, $f_A[a\,|\bar{D}]$, from Eqs. (12) and (13).[12]

As indicated by Eq. (3) crack size is a function of the parameters a_i and t_i and the time of inspection t. As a result Eqs. (12) and (13) are also dependent on these quantities. Since inspections were carried out at irregular intervals the above relations could only be utilized if detected cracks were mathematically grown to a common inspection time. Alternatively Eqs. (12-14) can be transformed to yield density functions of crack initiation times, t_i, at a common crack size, $a_i=0.03$ in. (0.76 mm).

IV. Distributions of Detected and Undetected Crack Initiation Times

Equations (12) and (13) may be rewritten in terms of crack initiation times, t_i as

$$f_{T_i}[t_i\,|D]=k_3P[D\,|t_i]f_{T_i}(t_i) \tag{15}$$

and

$$f_{T_i}[t_i\,|\bar{D}]=k_4(1-P[D\,|t_i\,])f_{T_i}(t_i) \tag{16}$$

where $f_{T_i}[t_i\,|D]$ and $f_{T_i}[t_i\,|\bar{D}]$ are the density functions of detected and undetected crack initiation times, $P[D\,|t_i]$ is the likelihood function for detection, and $f_{T_i}(t_i)$ is the "prior" density function of all crack initiation times.

$P[D\,|t_i]$ is derived from Eq. (10) by substitution of Eq. (3). Hence

$$P[D\,|t_i]=1-\exp\left\{-\left[(1/\bar{\beta}^{\bar{\alpha}})\left(\bar{C}^*(1-n^*)(t-t_i)+a_i^{(1-n^*)}\right)^{\bar{\alpha}/(1-n^*)}\right]\right\} \tag{17}$$

As a consequence the density functions of detected and undetected initiation times are as expected functions of the inspection time while the density function of all initiation times, $f_{T_i}(t_i)$, is independent of when the inspection is carried out.

It should be noted that Eq. (17) becomes equal to unity when

$$(t-t_i)=1-a_i^{(1-n^*)}/\bar{C}^*(1-n^*) \tag{18}$$

because within this time interval a crack will grow to infinite length.

The distributions of t_i described in Section II were derived from long-term records, in some instances covering 7000 h of flying time. It is therefore not unreasonable to assume that the great majority of cracks have grown to inspectable size and that distributions of crack initiation times derived from these by backward extrapolation are good approximations of the total initiation time distributions. Such an assumption is conservative because long initiation times (small cracks) make up the majority of still undetected initiations.

With this assumption it is possible to determine the density functions of discovered and undiscovered initiation times for an inspection carried out earlier in the life of the structure.

For specified inspection time, t, crack growth parameters C^* and n^* and $a_i=0.03$ in., Eqs. (17) and (9) substituted into Eqs. (15) and (16) yield the density functions of detected and undetected initiation times. As in the case of detected and undetected cracks the reciprocal normalizing coefficients add to unity

$$1/k_3+1/k_4=1 \tag{19}$$

Table 1 Normalizing constants and percentage of detected and undetected crack initiation times

Inspection time t,h	k_3	% Detected	k_4	% Undetected
2500	16.60	6.02	1.06	93.98
4000	1.60	62.56	2.67	37.44
5000	1.01	98.56	69.34	1.44

and indicate the fractions of discovered and undiscovered initiation times.

The results of such calculations are shown for wing location 76-92, on aircraft flown by the Tactical Air Command in Figs. 3-5 for inspection times $t = 2500$, 4000, and 5000 h. The normalizing coefficients, k_3 and k_4, are presented in Table 1.

It should be noted that the density functions for detected and undetected initiation times have been divided by their respective normalizing constants so that the areas under the two curves are proportional to the fractions of discovered and undiscovered initiation times (Table 1) and the two areas sum up to the area of the density function for all initiation times.

An examination of Figs. 3-5 indicates an expected trend. If inspection is performed too early, the majority of initiation times will remain undetected while more and more crack initiation times are found at later inspections.

The aforegoing discussion has utilized long-term inspection data to derive the prior distribution of crack initiation times. Such records are however usually unavailable for aircraft in service. On the contrary at a particular inspection time the distribution of observed crack initiation times is actually the posterior distribution, $f_{T_i}(t_i|D)$, while the prior density, $f_{T_i}(t_i)$, of all initiation times and the density function of undetected initiation times, $f_{T_i}(t_i|D\bar{d})$, are unknown.

Because the latter two functions are of primary interest Eqs. (15) and (16) may be rewritten using Eq. (19) as

$$f_{T_i}(t_i) = \frac{f_{T_i}[t_i|D]}{k_3 P[D|t_i]} \tag{20}$$

and

$$f_{T_i}(t_i|\bar{D}) = \frac{(1-P[D|t_i])f_{T_i}[t_i|D]}{(k_3-1)P[D|t_i]} \tag{21}$$

These equations would predict the density functions of all crack initiation times as well as that of the undetected times provided that inspections are carried out at approximately the same number of flight hours on all aircraft in a fleet.

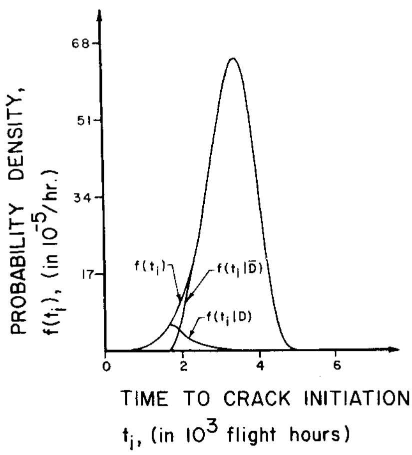

Fig. 3 Density functions of crack initiation times at an inspection time, $t = 2500$ h.

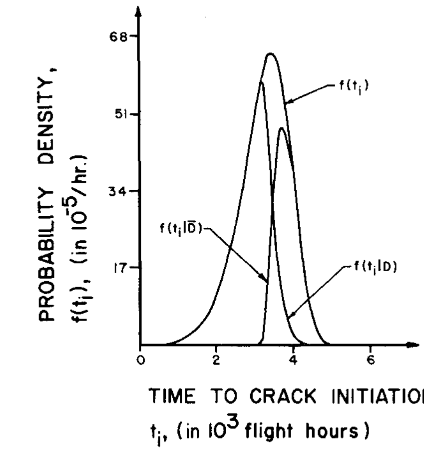

Fig. 4 Density functions of crack initiation times at an inspection time, $t = 4000$ h.

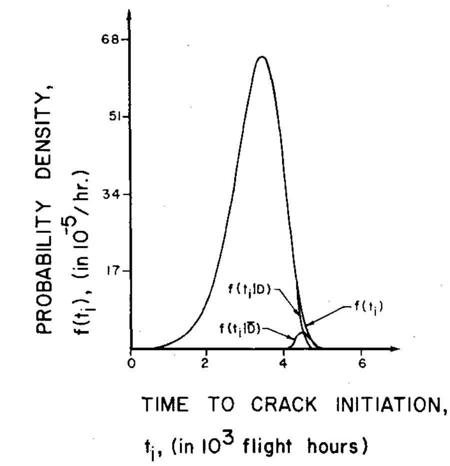

Fig. 5 Density functions of crack initiation times at an inspection time, $t = 5000$ h.

V. Conclusions

The Bayesian approach presented here illustrates a method for estimating the proportion of undiscovered crack initiation times at various inspections and indicates the influence of the reliability of crack detection methodology particularly during early inspections when cracks are small.

Because the information concerning the reliability of the particular method of nondestructive examination used for the C-130 aircraft was not available an assumed likelihood function was used. For future systems the documentation of NDE reliability should become an integral part of service inspection.

A similar approach may be used to determine the fraction of undetected crack sizes at a particular inspection time. For this purpose cracks would have to be analytically grown to common inspection times and their distributions analyzed. Such work may be carried out in the future.

The present analyses have utilized average values of the crack propagation parameter C^*. The statistical distributions of the parameter have not been calculated. Calculations, with C^* as a random variable, may be performed at a later date.

Acknowledgments

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